Heat eq

$u_t = k u_{xx} \qquad 0<x<L \qquad t>0$

$u(0,t) = u(L,t) = 0$

or $u_x(0,t) = u_x(L,t) = 0$

$u(x,0) = f(x)$

separation of variables: $u(x,t) = X(x)\,T(t)$

$u_t = X T' \qquad u_{xx} = X'' T$

$u_t = k u_{xx} \to X T' = k X'' T$

$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \qquad (\lambda \text{ is ok, too})$

$X'' + \lambda X = 0$

$X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$

if $u(0,t)=u(L,t)=0 \rightarrow X(0)=X(L)=0$

$$0 = A$$

$$0 = B\sin(\sqrt{\lambda}L) \qquad B \neq 0,\ \lambda > 0$$

$$\sin(\sqrt{\lambda}L) = 0$$

$$\sqrt{\lambda}L = n\pi \qquad n = 1,2,3,\dots$$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}} \qquad \boxed{X_n = \sin\left(\frac{n\pi}{L}x\right)}$$

if $u_x(0,t)=u_x(L,t)=0 \rightarrow X'(0)=X'(L)=0$

$$X'(x) = -\sqrt{\lambda}A\sin(\sqrt{\lambda}x) + \sqrt{\lambda}B\cos(\sqrt{\lambda}x)$$

$$0 = \sqrt{\lambda}B \rightarrow B = 0$$

$$0 = -\sqrt{\lambda}A\sin(\sqrt{\lambda}L) \rightarrow \sin(\sqrt{\lambda}L) = 0 \rightarrow \boxed{\lambda_n = \frac{n^2\pi^2}{L^2}} \qquad n = 1,2,3,\dots$$

$$\boxed{X_n = \cos\left(\frac{n\pi}{L}x\right)}$$

if $\lambda = 0$, $\quad X'' + \lambda X = 0 \rightarrow X'' = 0$

$$X = Ax + B$$

$$X'(0) = X'(L) = 0 \longrightarrow A = 0 \rightarrow \boxed{X_0 = 1} \quad \boxed{\lambda_0 = 0}$$

if $u(0,t) = 0$

$u_x(L,t) = 0$

$\downarrow$

$X(0) = 0$

$X'(L) = 0$

$$X'' + \lambda X = 0$$

if $\lambda > 0$, $\quad X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$$X(0) = 0 \rightarrow A = 0 \quad X = B\sin(\sqrt{\lambda}x) \quad X' = \sqrt{\lambda}B\cos(\sqrt{\lambda}x)$$

$$X'(L) = 0 \rightarrow 0 = \sqrt{\lambda}B\cos(\sqrt{\lambda}L) \quad B \neq 0$$

$$\cos(\sqrt{\lambda}L) = 0$$

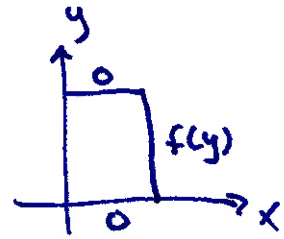
$$\sqrt{\lambda}L = \frac{(2n+1)}{2}\pi \quad n = 0, 1, 2, 3,$$

$$= \frac{(2n-1)}{2}\pi \quad n = 1, 2, 3, \ldots$$

$$\boxed{\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}} \quad n = 1, 2, 3, \ldots \quad \boxed{X_n = \sin(\sqrt{\lambda_n}x)}$$

Laplace's eq

$u_{xx} + u_{yy} = 0 \qquad 0<x<a \qquad 0<y<b$



$u(x,0) = 0$

$u(x,b) = 0$

$u_x(0,y) = 0$

$u(a,y) = f(y)$

$X''Y + XY'' = 0$

$$\underbrace{\frac{X''}{X} = -\frac{Y''}{Y} = +\lambda}$$

since y has 0 BC

we want to solve $Y'' + \lambda Y = 0$

$Y'' + \lambda Y = 0 \qquad Y(0) = Y(b) = 0$

$\boxed{\lambda_n = \frac{n^2\pi^2}{b^2}} \qquad n = 1, 2, 3, \ldots$

$\boxed{Y_n = \sin\left(\frac{n\pi}{b} y\right)}$

$$\frac{X''}{X} = \lambda = \frac{X''}{X} = \frac{n^2\pi^2}{b^2}$$

$$X'' - \frac{n^2\pi^2}{b^2} X = 0 \qquad X'(0) = 0$$

$$X = A\cosh\left(\frac{n\pi}{b}x\right) + B\sinh\left(\frac{n\pi}{b}x\right)$$

$$X' = A\cdot\frac{n\pi}{b}\sinh\left(\frac{n\pi}{b}x\right) + B\cdot\frac{n\pi}{b}\cosh\left(\frac{n\pi}{b}x\right)$$

$$0 = B \qquad \boxed{X_n = \cosh\left(\frac{n\pi}{b}x\right)}$$

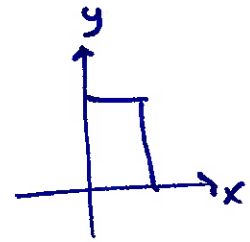
$$u(x,y) = \sum_{n=1}^{\infty} C_n\cosh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$u(a,y) = f(y) = \sum_{n=1}^{\infty}\left[C_n\cosh\left(\frac{n\pi a}{b}\right)\right]\sin\left(\frac{n\pi}{b}y\right)$$

$$C_n\cosh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy$$

2D Wave

$$u_{tt} = a^2(u_{xx} + u_{yy}) \qquad 0<x<a \qquad 0<y<b$$



$$u(x,0)=0 \qquad u(x,y,0)=f(x)$$

$$u(x,b)=0 \qquad u_t(x,y,0)=g(x)$$

$$u(a,y)=0$$

$$u(0,y)=0$$

$$u(x,y,t) = XYT$$

$$XYT'' = a^2(X''YT + XY''T)$$

$$\frac{T''}{T} = a^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right)$$

$$\frac{T''}{a^2T} = \frac{X''}{X} + \frac{Y''}{Y}$$

$$\frac{X''}{X} = \frac{T''}{a^2T} - \frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \qquad X(0) = X(a) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{a^2} \qquad X_n = \sin\left(\frac{n\pi}{a}x\right)$$

$$\frac{T''}{a^2T} - \frac{Y''}{Y} = -\lambda \qquad \frac{Y''}{Y} = \frac{T''}{a^2T} + \lambda = -\mu$$

$$Y'' + \mu Y = 0 \qquad Y(0) = Y(b) = 0$$

$$\mu_m = \frac{m^2\pi^2}{b^2} \qquad Y_m = \sin\left(\frac{m\pi}{b} y\right)$$

$$\frac{T''}{a^2T} = -(\lambda + \mu)$$

$$T'' + a^2(\lambda + \mu)T = 0$$

$$T = A\cos\left(a\sqrt{\lambda+\mu}\, t\right) + B\sin\left(a\sqrt{\lambda+\mu}\, t\right)$$

$\vdots$

T_{nm}

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} T_{nm} X_n Y_m$$